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Fractal Weyl law for quantum fractal eigenstates

D. L. Shepelyansky

Laboratoire de Physique Théorique, UMR 5152 du CNRS, Université Toulouse III, 31062 Toulouse, France

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The properties of the resonant Gamow states are studied numerically in the semiclassical limit for the quantum Chirikov standard map with absorption. It is shown that the number of such states is described by the fractal Weyl law, and their Husimi distributions closely follow the strange repeller set formed by classical orbits nonescaping in future times. For large matrices the distribution of escape rates converges to a fixed shape profile characterized by a spectral gap related to the classical escape rate.

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The Weyl law [1] gives a fundamental link between the properties of quantum eigenstates in closed Hamiltonian systems, the Planck constant $\hbar$, and the classical phase space volume. The number of states in this case is determined by the system dimension d, and the situation is now well understood on both mathematical and physical grounds [2,3]. Surprisingly, only recently has it been realized that the case of nonunitary operators describing open systems in the semiclassical limit has a number of new interesting properties, and the concept of the fractal Weyl law has been introduced to describe the dependence of the number of resonant Gamow eigenstates on $\hbar$ [4,5]. The Gamow eigenstates find important applications for decay of radioactive nuclei [6], quantum chemistry reactions [7], chaotic scattering [8], and microlasers with chaotic resonators [9–11]. Thus the understanding of their properties in the semiclassical limit represents an important challenge.

According to the fractal Weyl law [4,5], the number of Gamow eigenstates N_γ that have escape rates γ in a finite bandwidth $0 \leq \gamma \leq \gamma_b$ scales as

$$N_\gamma \propto \hbar^{-(d-1)}, \tag{1}$$

where d is the fractal dimension of a classical strange repeller formed by classical orbits nonescaping in future (or past) times. By numerical simulations it has been shown that the law (1) works for a three-disk system [5] and quantum chaos maps with absorption [12,13] at specific values of d. Recent mathematical results for open quantum maps are presented in [14]. The law (1) for open systems with a fractal dimension $d<2$ leads to a striking consequence: only a relatively small fraction of eigenstates $\mu \sim N_\gamma / N \propto \hbar^{(2-d)}$ have finite values of γ, while almost all eigenstates of the matrix operator of size $N \propto 1/\hbar$ have infinitely large γ. The latter states are associated [12] with classical orbits which escape from the system after the Ehrenfest time [15]. The former states with finite γ are related to the classical fractal repeller and have been named quantum fractal eigenstates due to the fractal structure of their Husimi distributions closely following the classical fractal [16]. The semiclassical description of probability density for such states has been derived recently in [17].

In view of the recent results described above, I study numerically a simple model of the quantum Chirikov standard map (kicked rotator) with absorption introduced in [18] which allows continuous variation of the fractal dimension of the classical strange repeller. In this way the fractal Weyl law (1) is verified in the whole interval $1 \leq d \leq 2$. The model also allows us to establish the limiting semiclassical distribution over escape rates γ and find its links with the fractal properties of the classical strange repeller. The Chirikov standard map is a generic model of chaotic dynamics, and it finds applications in various physical systems including magnetic mirror traps, accelerator beams, and Rydberg atoms in a microwave field [19–22]. The quantum model has been built up in experiments with cold atoms [23]. Thus the results obtained for this model should be generic and should find applications for various systems.

The quantum dynamics of the model is described by the evolution matrix

$$\bar{\psi} = \hat{U}\psi = \hat{P} e^{-iT\hat{n}^2/4} e^{-ik\cos\hat{\theta}} e^{-iT\hat{n}^2/4}\psi, \tag{2}$$

where $\hat{n} = -i\partial/\partial\theta$ and the operator $\hat{P}$ projects the wave function to the states in the interval $[-N/2, N/2]$. The semiclassical limit corresponds to $k \to \infty$, $T \to 0$ with the chaos parameter $K = kT = \text{const}$ and absorption boundary $a = N/k = \text{const}$. Thus N is inversely proportional to the effective Planck constant $T = \hbar_{\text{eff}}$; it gives the number of quantum eigenstates and the number of quantum cells inside the classical phase space. The classical dynamics is described by the Chirikov standard map [19,20] in its symmetric form:

$$\bar{n} = n + k\sin\left(\theta + \frac{Tn}{2}\right), \quad \bar{\theta} = \theta + \frac{T}{2}(n + \bar{n}). \tag{3}$$

Physically, the map describes a free particle propagation in the presence of periodic kicks with period T (e.g., kicks of the optical lattice in [23]). In this model all trajectories (and quantum probabilities) escaping the interval $[-N/2, N/2]$ are absorbed and never return back. It is convenient to fix $K=7$ so that the phase space has no visible stability islands for $a \leq 6$ [24]. Then for the classical dynamics the probability $P(t)$ to stay inside decays exponentially with time as $P(t) \sim \exp(-\gamma_c t)$ [18,25], where γ_c is the classical escape rate and t is measured in the number of map iterations. For large values of a the spreading goes in a diffusive way and $t_c = 1/\gamma_c \sim N^2/D \approx 2a^2$ where $D \approx k^2/2$ is the diffusion rate for $K \gg 1$. The independence of t_c of N implies $a = N/k = \text{const}$. The quantum operator (2) can be considered as a simplified model of chaotic microlasers, where all rays with orbital mo-

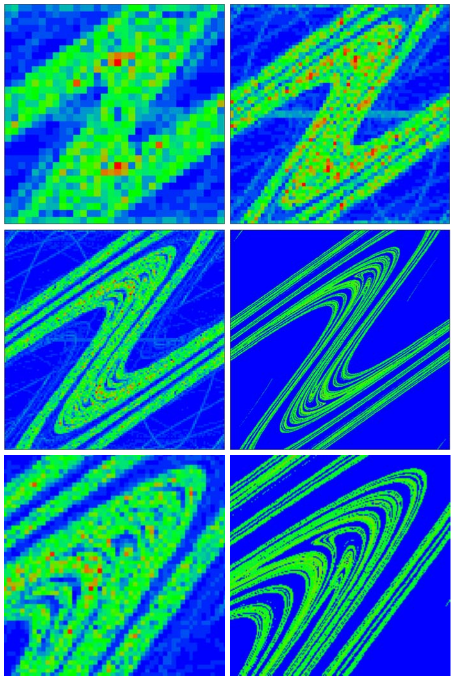

FIG. 1. (Color online) Husimi functions of quantum fractal eigenstates with minimal value of γ at $N=1025$ (top left), 4097 (top right), and 16 349 (middle left) and the density plot of the classical strange chaotic repeller formed by orbits nonescaping forward in time (middle right); the two bottom panels show zooms for the two middle panels, respectively. Here $a=N/k=2$, $K=kT=7$, and the box counting dimension of the repeller is $d=1.7230$. In the top and middle panels $0\leq\theta<2\pi$, $-N/2\leq n\leq N/2$; density is proportional to color with red (gray) for maximal density and blue (black) for zero density.

menta below some critical value determined by the refraction index escape from a microcavity [9–11].

The right eigenstates $\psi_n^{(m)}$ and eigenvalues $\lambda_m=\exp(-i\epsilon_m-\gamma_m/2)$ of the evolution operator $\hat{U}$ are determined numerically by direct diagonalization up to a maximal value $N=22\,001$ (only states symmetric in n are considered). The Husimi distribution [26], obtained from smoothing of a Wigner function on a Planck constant scale, is shown in Fig. 1 for eigenstates with minimal γ_m at different values of N at $a=2$. With the increase of N, the Husimi distribution converges to a fractal set, which is very similar to the classical strange repeller formed by classical orbits never escaping in the future. The classical repeller is obtained by iterating up to 3×10^9 classical trajectories homogeneously distributed in the whole phase space at $t=0$. The classical remaining probability $P(t)$ decays exponentially with $\gamma_c=0.2702\pm0.0011$ and the computation of the box counting dimension [20,21] of the strange repeller gives $d=1.7230\pm0.0085$. According to [8,20,21], the information dimension d_1 of the repeller can be expressed as $d_1=2-\gamma_c/\Lambda$, where Λ is the Lyapunov exponent. For large a and small γ_c it can be expressed via its value for the Hamiltonian dynamics on a torus where $\Lambda\approx\ln(K/2)=1.2527$ (for $K=7$) [19]. This gives $d_1=1.7843$, which is rather close to the numerical value of the box counting dimension d (usually these two dimensions are rather close and, contrary to [13], I will not make a distinction between them). However, for smaller values of a the relation $\Lambda=\ln(K/2)$ is no longer valid. To have Λ for all values of a, its value is computed numerically following the approximation used in [19]: $\Lambda=\langle|\ln[K\cos(\theta+Tn/2)]|\rangle$, where the average $\langle\cdots\rangle$ is done over the orbits on the repeller. In this way Λ varies in the interval $1.913\leq\Lambda\leq1.294$ for $0.7\leq a\leq6$ ($\Lambda=1.363$ and $d_1=1.801$ at $a=2$).

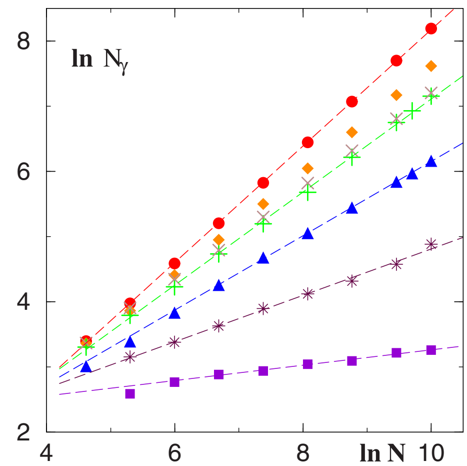

FIG. 2. (Color online) Dependence of the integrated number of states N_γ with escape rates $\gamma\leq\gamma_b=8/a^2$ on matrix size N. Symbols show numerical data for various values of absorption border $a=N/k$: 4 (circles), 2.5 (diamonds), 2 (+), 1.5 (×), 1 (triangles), 0.8 (asterisks), and 0.7 (squares). Dashed lines show algebraic fits $N_\gamma\propto N^\nu$ with the fractal Weyl exponent $\nu=0.8930\pm0.0028$ ($a=4$), $\nu=0.7129\pm0.0073$ ($a=2$), $\nu=0.5697\pm0.0042$ ($a=1$), $\nu=0.3559\pm0.0094$ ($a=0.8$), and $\nu=0.1175\pm0.0069$ ($a=0.7$; here $\gamma_b=4/a^2$). Logarithms are natural.

To check the validity of the fractal Weyl law (1) for various d, the absorption border a is varied in the interval $0.7\leq a\leq6$ so that the classical fractal dimension and decay rate vary in the intervals $0.9976\pm0.0060\leq d\leq1.9367\pm0.0067$ and $1.6349\pm0.0135\leq\gamma_c\leq0.0592\pm0.0003$. In the quantum case the number of states N_γ is computed in the bandwidth $0\leq\gamma\leq\gamma_b$ with $\gamma_b=8/a^2$. In this way $\gamma_b>\gamma_c$ and the band contains a large fraction of fractal eigenstates. To improve the statistics, N_γ is averaged over N_r cases with slightly different values of $k\pm\delta k$ with $\delta k\leq2$. Such a small variation of k does not affect the semiclassical properties but allows one to improve the statistical accuracy. The number of realizations varied from $N_r=40$ at $N=101$ to $N_r=1$ at $N=22\,001$. The dependence of the integrated number of states N_γ on $N\propto1/\hbar$ is shown in Fig. 2. The fit $N_\gamma\propto N^\nu$ allows determination of the exponent ν, which according to (1) should satisfy the relation $\nu=d-1$. It is important to stress that $N_\gamma\ll N/2$ at $a\leq2$, so that the main part of the $(N+1)/2$ eigenvalues has enormously large $\gamma\gg\gamma_b$.

The dependence of ν on d is shown in Fig. 3. The law (1) is well satisfied for fractal dimensions $1\leq d<2$. Certain deviations for d close to 1 should be attributed to rather small values of N_γ (e.g., $N_\gamma=26$ at $a=0.7$ and $N=22\,001$) so that even larger N values are required to see the asymptotic behavior. The relation $\nu=1-\gamma_c/\Lambda$ also works rather well even

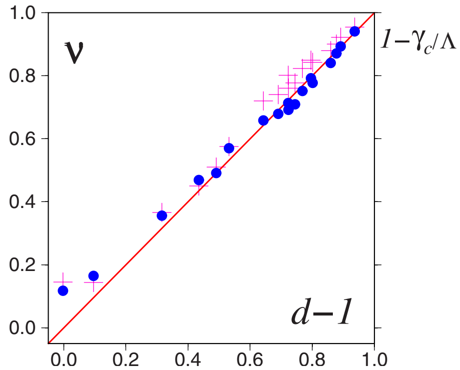

FIG. 3. (Color online) Dependence of the fractal Weyl exponent ν on the fractal box counting dimension d: full circles show numerical data, the straight line shows the fractal Weyl law (1) $\nu=d-1$, pluses show $1-\gamma_c/\Lambda$ versus $d-1$, which should follow the relation $d_1-1=1-\gamma_c/\Lambda$, where Λ is the Lyapunov exponent computed approximately (see text). Here $0.7\leq a\leq 6$, $0.1175\leq\nu\leq 0.9402$, $0.9976\leq d\leq 1.9367$.

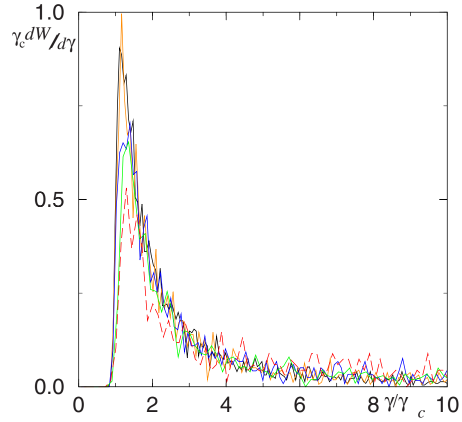

FIG. 5. (Color online) Same distribution as in Fig. 4 drawn for various absorption boundaries a at $N=22001$. Here a is 4 ($\gamma_c=0.1019$, $N_\gamma=3607$, black curve); 2.5 [$\gamma_c=0.2063$, $N_\gamma=2032$, orange (gray curve); 2 ($\gamma_c=0.2702$, $N_\gamma=1278$, blue (black curve); 1.5 ($\gamma_c=0.2961$, $N_\gamma=1342$, green (gray curve); 1 ($\gamma_c=0.6967$, $N_\gamma=472$, red (gray dashed curve)].

though the Lyapunov exponent Λ should probably be computed in a more exact way for small values of $a\sim 0.7$. Thus, the data of Fig. 3 confirm the validity of the fractal Weyl law in the whole available interval of fractal dimensions.

In addition to the integrated characteristic (1) it is interesting to consider the differential distribution $dW/d\gamma$, which determines the number of states in the interval $d\gamma$ at given γ. The evolution of distributions $dW/d\gamma$ with the growth of N is shown in Fig. 4. The data clearly show that in the semiclassical limit $dW/d\gamma$ converges to a certain limiting distribution independent of N. This effect has been noticed already in earlier studies [18] where mainly the diffusive limit

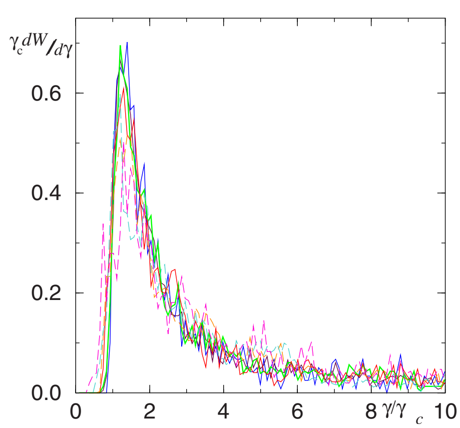

FIG. 4. (Color online) Dependence of the distribution $\gamma_c dW/d\gamma$ on the rescaled escape rate γ/γ_c for different values of N at $a=2$. Here $\gamma_c=0.2702$, $d=1.7230$, the probability $dW/d\gamma$ is normalized to unity in the interval γ_b, and N is 22 001 [blue (black) full curve $N_\gamma=1278$], 12 801 [maroon (gray) full curve $N_\gamma=1022$], 6401 [green (gray) full curve $N_\gamma=500$], 3201 [red (gray) full curve $N_\gamma=293$], 1601 [orange (gray) dashed line $N_\gamma=181$], 801 [turquoise (gray) dashed lines $N_\gamma=114$], and 401 [magenta (gray) dashed line $N_\gamma=68.7$].

with $a=10$ and $\gamma_c\gg\Lambda$ was considered. In such a case the dimension is very close to the integer value $d=2$ and therefore the fractal dependence (1) was missed in [16,18] even though the fractal structure of the eigenstates was clearly detected [16]. In the diffusive case $d\approx 2$, one has $dW/d\gamma \propto 1/\gamma^{3/2}$ for $\gamma>\gamma_c$, which is explained by simple estimates [18] and more rigorous analytical treatment [27]. When the fractal dimension d is noticeably less than 2, then $\gamma_c\sim\Lambda$, and the diffusive approximation is no longer valid. A distinctive feature of the distribution in this case is the gap in the distribution $dW/d\gamma$, which is zero for $\gamma<\gamma_c$, a sharp peak at $\gamma=\gamma_c$, followed by a smooth drop at $\gamma>\gamma_c$ (this drop is compatible with the dependence $1/\gamma^{3/2}$).

These properties of the distribution $dW/d\gamma$ remain essentially the same when γ_c is changed by a factor 3.5, as shown in Fig. 5. Indeed, the shape of the distribution varies very little for $1.5\leq a\leq 4$ and becomes broader only at $a<1.5$. The latter case, however, has relatively small statistics N_γ, and probably larger N should be used to reach a limiting distribution for $a<1.5$. It is interesting to note that $dW/d\gamma$ has certain similarities with the Wigner proper time distribution discussed in [28].

In conclusion, the data obtained confirm the validity of the fractal Weyl law for all fractal dimensions in the interval $1\leq d\leq 2$. They show the existence of a limiting distribution of the Gamow resonances $dW/d\gamma$ which has a gap of size γ_c, above which the distribution has a sharp peak (see Figs. 4 and 5). Thus the classical decay rate γ_c essentially determines the quantum decay rates on the quantum fractal corresponding to the classical strange repeller with orbits never escaping in future times (Fig. 1). The analytical computation of the limiting distribution $dW/d\gamma$ still remains an open problem. It is possible that the analytical methods pushed forward recently [17] will allow progress to be made in this direction. Also, it would be interesting to check the validity of the fractal Weyl law for dimensions $d>2$. In such a case it is natural to expect that $N_\gamma \propto \hbar^{-\nu}$ with $\nu = d - n_f$, where d is the fractal dimension of the classical strange repeller and n_f is the number of degrees of freedom (in the present model $n_f=1$, $d \leq 2$).

At present the properties of large nonunitary matrices find important applications in various areas, including search on the Internet [29,30], and it is possible that the fractal quantum eigenstates may have certain applications there, since they give an example of important nontrivially connected fractal sets of small measure.

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